Zoom Class 28/4/2020

Centre of Curvature

If $C(\alpha, \beta)$ then α and β are its co-ordinates then

$$\boxed{\begin{aligned} \alpha &= x - \frac{y_1(1+y_1^2)}{y_2} \\ \text{and} \quad \beta &= y + \frac{1+y_1^2}{y_2} \end{aligned}}$$

where x, y are co-ordinates of any point P on the given curve

Also Equation of circle of curvature is

$$(x-\alpha)+(y-\beta)^2 = \rho^2$$

where x, y are co-ordinates of point P on curve.
α, β are co-ordinates of centre of curvature and ρ is radius of curvature.

Evolute – It is the locus of centre of curvature of a curve and also the given curve is called involute of its evolute.

To find the equation of evolute we eliminate x and y from the co-ordinates of α, β from the centre of curvature

(i) The normal at any point of the curve is the tangent to the evolute at the centre of curvature for that point.

(ii) The difference between the radii of curvature at any two points on a curve is equal to the length of the arc of the evolute between the corresponding points.

Ques For the curve $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$
Prove that $\alpha = a(\theta + \sin\theta)$, $\beta = -a(1-\cos\theta)$
where (α, β) are co-ordinates of the centre of curvature.

Soln Since $x = a(\theta - \sin\theta)$ ~~and~~

$$\Rightarrow \frac{dx}{d\theta} = a(1-\cos\theta) \text{ ——— } ①$$

$$y = a(1-\cos\theta) \Rightarrow \frac{dy}{d\theta} = a\sin\theta \quad ②$$

23 SATURDAY
Week 21 ■ 144-222

$\therefore$ ① & ② implies

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

$$= \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \cot\left(\frac{\theta}{2}\right) \text{ — } ③$$

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2\left(\frac{\theta}{2}\right) \cdot \frac{1}{2} \cdot \frac{d\theta}{dx}$$

$$= -\operatorname{cosec}^2\left(\frac{\theta}{2}\right) \cdot \frac{1}{2} \cdot \frac{1}{a(1-\cos\theta)}$$

SUNDAY 24

$$= \frac{1}{2a} \frac{\operatorname{cosec}^2(\theta/2)}{2\sin^2(\theta/2)} = \frac{1}{4a} \operatorname{cosec}^4 \theta/2 \text{ — } ④$$

(3) & (4) implies

Now $\dfrac{1+y_1^2}{y_2} = \dfrac{1+\cot^2(\theta/2)}{-\frac{1}{4a}\operatorname{cosec}^4(\theta/2)}$

$$= -4a\left\{\frac{\operatorname{cosec}^2(\theta/2)}{\operatorname{cosec}^4(\theta/2)}\right\} = -4a\sin^2\left(\frac{\theta}{2}\right) \quad (5)$$

Now using (5) we get

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2} = a(\theta - \sin\theta) - \cot\left(\frac{\theta}{2}\right)\left(-4a\sin^2\left(\frac{\theta}{2}\right)\right)$$

$$= a\left\{(\theta - \sin\theta) + 4\cot\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\theta}{2}\right)\right\}$$

$$= a\{\theta - \sin\theta + 4\cos(\theta/2)\sin(\theta/2)\}$$

$$= a\{\theta - \sin\theta + 2\sin\theta\}$$

$$= a\{\theta + \sin\theta\} \quad (6)$$

$$\beta = y + \frac{1+y_1^2}{y_2}$$

$$= a(1-\cos\theta) - 2a(1-\cos\theta)$$

$$= (1-\cos\theta)\{a - 2a\}$$

$$= -a(1-\cos\theta) \quad (7)$$

Eqn (6) (7) give co-ordinates of centre of curvature

Proved

Ques) Find the co-ordinates of the centre of curvature for any point (x, y) on the parabola $y^2 = 4ax$. Also, find the equation of the evolute of the parabola.

Soln) $\because$ The equation of the parabola $y^2 = 4ax$.

$$\Rightarrow 2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y} \quad \text{—} (1)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2a}{y^2}\cdot\frac{dy}{dx} = -\frac{2a}{y^2}\cdot\frac{2a}{y} = -\frac{4a^2}{y^3} \quad \text{—} (2)$$

$\because (\alpha, \beta)$ are co-ordinates of centre of curvature.

28 THURSDAY Week 22 149-217

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\Rightarrow \alpha = x - \frac{(2a/y)(1+4a^2/y^2)}{(-4a^2/y^3)}$$

$$= x + \frac{(2a)(y^2+4a^2)\cdot\cancel{y^3}}{\cancel{y^3}(-\cancel{4}a^{\cancel{2}})} = x + \frac{(y^2+4a^2)}{2a}$$

$$= x + \frac{(4ax+4a^2)}{2a} = x + \frac{\cancel{4}^{2}a(x+a)}{\cancel{2a}}$$

$$= x + 2(x+a) = x + 2x + 2a = (3x+2a) \quad \text{—} (3)$$

Also
$$\beta = y + \frac{1+y_1^2}{y_2} = y + \frac{1+(4a^2/y^2)}{(-4a^2)/y^3}$$

$$= y + \frac{(y^2+4a^2)}{\cancel{y^2}}\cdot\frac{\cancel{y^3}\,y}{-4a^2}$$

$$= y + \frac{y(y^2+4a^2)}{-4a^2}$$

$$\beta = y - \frac{y(4ax + 4a^2)}{4a^2}$$

$$= y - \frac{4ay(x+a)}{4a^2} = y\left\{1 - \left(\frac{x+a}{a}\right)\right\}$$

$$= y\left\{\frac{\cancel{a} - x - \cancel{a}}{a}\right\} = y\left(-\frac{x}{a}\right) = -\frac{1}{a}x \cdot \sqrt{4ax}$$

$$= -\frac{2}{\sqrt{a}}x^{1+1/2} = -\frac{2}{\sqrt{a}}x^{3/2} \longrightarrow (4).$$

Now (1) $\Rightarrow \alpha - 2a = 3x$.

$\Rightarrow (\alpha - 2a)^3 = 27x^3 \longrightarrow (5)$

And (2) $\Rightarrow \beta\sqrt{a} = -2x^{3/2}$

$\Rightarrow \beta^2 a = 4x^3 \longrightarrow (6)$

(5) divided by (6) we have.

$$\frac{(\alpha - 2a)^3}{\beta^2 a} = \frac{27}{4}$$

$\Rightarrow 4(\alpha - 2a)^3 = 27\beta^2 a$ is the required Equation of envolute.

Ans